

Backward stochastic viability property with jumps and applications to the comparison theorem for multidimensional BSDEs with jumps

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Abstract

In this paper, we study conditions under which the solutions of a backward stochastic differential equation with jump remains in a given set of constraints. This property is the so-called "viability property". As an application, we study the comparison theorem for multidimensional BSDEs with jumps.

Keywords: Viability property; BSDE; Comparison theorem.

1 Introduction

Viability properties for stochastic differential equations (SDEs) and inclusions had been introduced and studied by Aubin-Da Prato in [2], [3] and [4]. The key point of their work was a "stochastic tangent cone" which generalized the well-known Bouligand's contingent cone used for deterministic systems. Buckdahn-Peng-Quincampoix-Rainer [6] used a new method to get the necessary and sufficient condition for the viability property of SDEs with control. They related viability with a kind of optimal control problem and applied the comparison theorem for viscosity solutions to some H-J-B equation. In 2000, renewed interests arouse in the study of viability for backward stochastic differential equations (BSDEs) in the paper of Buckdahn-Quincampoix- Răşcanu [7]. Their approach differed from the above mentioned methods and based on the convexity of the distance function of K (when K is a closed convex set). This enabled them to deduce some condition in differential form on the distance function of K which is necessary as well as sufficient.

In the present paper, we shall adopt the approach of Buckdahn et al. [7] to study the backward stochastic viability property (BSVP) with jumps.

As we know, the comparison theorem and the converse comparison theorem for real-valued BSDEs with or without jumps had been studied by many mathematicians. We refer to Peng [13], Pardoux-Peng [12], El Karoui-Peng-Quenez [9], Coquet-Hu-Memin-Peng [8], Barles-Buckdahn-Pardoux [5], Wu [15], Royer [14] and their references. In 2006, Hu-Peng [10] first use the BSVP to study the comparison theorem for multidimensional BSDEs and get necessary and sufficient condition.

So We can apply our results of the BSVP with jumps to study the comparison theorem for multidimensional BSDEs with jumps combining the approach used in Hu-Peng [10].

This paper is organized as follows: In the next section, we state some basic assumptions and basic estimates for BSDEs with jumps. And then we state the main result of the paper on the BSVP. In Section 3, we apply our main result to the comparison theorem for BSDEs with jumps. Finally, in Appendix, we give the proof of our main result.

2 The BSVP with jumps in closed sets

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ be a complete stochastic basis such that \mathcal{F}_0 contains all P -null elements of \mathcal{F} , and $\mathcal{F}_{t+} := \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t, t \geq 0$, and $\mathcal{F} = \mathcal{F}_T$, and suppose that the filtration is generated by the following two mutually independent processes:

- (i) a d -dimensional standard Brownian motion $(W_t)_{0 \leq t \leq T}$, and
- (ii) a stationary Poisson random measure N on $(0, T] \times E$, where $E \subset \mathbb{R}^l \setminus \{0\}$, E is equipped with its Borel field \mathcal{B}_E , with compensator $\hat{N}(dtde) = dt n(de)$, such that $n(E) < \infty$, and $\{\tilde{N}((0, t] \times A) = (N - \hat{N})((0, t] \times A)\}_{0 \leq t \leq T}$ is an \mathcal{F}_t -martingale, for each $A \in \mathcal{B}_E$.

By $T > 0$ we denote the finite real time horizon. We define some spaces of processes. Let $\mathcal{S}_{[0, T]}^2$ denote the set of \mathcal{F}_t -adapted càdlàg m -dimensional processes $\{Y_t, 0 \leq t \leq T\}$ which are such that

$$\|Y\|_{\mathcal{S}_{[0, T]}^2} := (E[\sup_{0 \leq t \leq T} |Y_t|^2])^{\frac{1}{2}} < \infty.$$

Let $L_{[0, T]}^2(W)$ be the set of \mathcal{F} -progressively measurable $m \times d$ dimensional processes $\{Z_t, 0 \leq t \leq T\}$ which are such that

$$\|Z\|_{L_{[0, T]}^2(W)} := (E \int_0^T |Z_t|^2 dt)^{\frac{1}{2}} < \infty.$$

By $L^2_{[0,T]}(\tilde{N})$ we denote the set of mapping $U : \Omega \times [0, T] \times E \rightarrow R^m$ which are $\mathcal{P} \times \mathcal{B}_E$ measurable and such that

$$\|U\|_{L^2_{[0,T]}(\tilde{N})} := (E \int_0^T \int_E |U_t(e)|^2 n(de) dt)^{\frac{1}{2}} < \infty,$$

where \mathcal{P} denotes the σ -algebra of \mathcal{F}_t -predictable subsets of $\Omega \times [0, T]$. Finally we define

$$\mathcal{B}^2_{[0,T]} = \mathcal{S}^2_{[0,T]} \times L^2_{[0,T]}(W) \times L^2_{[0,T]}(\tilde{N}).$$

Consider the following BSDE with jump:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \tilde{N}(ds de), 0 \leq t \leq T, \quad (2.1)$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, P, R^m)$ is a given random variable and $f : \Omega \times [0, T] \times R^m \times R^{m \times d} \times L^2(E, \mathcal{B}_E, n; R^m) \rightarrow R^m$ a measurable function.

We suppose that there exists $L > 0$ such that

- (H1) $f(\cdot, \cdot, y, z, u)$ is progressively measurable, $f(\omega, \cdot, y, z, u)$ is continuous,
- (H2) $|f(t, y, z, u) - f(t, y', z', u')| \leq L(|y - y'| + |z - z'| + \|u - u'\|),$
- (H3) $\sup_{0 \leq t \leq T} |f(t, 0, 0, 0)| \in L^2(\Omega, \mathcal{F}_T, P, R^m),$

for all $0 \leq t \leq T$, $y, y' \in R^m$, $z, z' \in R^{m \times d}$, $u, u' \in L^2(E, \mathcal{B}_E, n; R^m)$, $P - a.s..$

Let's recall the existence and uniqueness result for BSDEs with jumps (see [5]):

Proposition 2.1. *Let (H1)-(H3) holds true. Then for any given $\xi \in L^2(\Omega, \mathcal{F}_T, P, R^m)$, there exists an unique triple $(Y, Z, U) \in \mathcal{B}^2_{[0,T]}$ which solves BSDE (2.1).*

Note that the notion of BSDE with jump generalizes the well-known martingale representation property. Indeed, in the particular case of $f = 0$ we have the following Lemma (see [11]):

Lemma 2.2. *For any $\xi \in L^2(\Omega, \mathcal{F}_T, P, R^m)$, there exist a unique $R(\xi)$ belonging to $L^2_{[0,T]}(W)$ and a $V \in L^2_{[0,T]}(\tilde{N})$ such that*

$$\xi = E(\xi) + \int_0^T R_s dW_s + \int_0^T \int_E V_s(e) \tilde{N}(ds de). \quad (2.2)$$

For completeness, we give the following definition of stochastic viability:

Definition 2.3. *Let K be a nonempty closed subset of R^m .*

(a) *A stochastic process $\{Y_t, t \in [0, T]\}$ is viable in K if and only if for P -almost $\omega \in \Omega$,*

$$Y_t(\omega) \in K, \quad \forall t \in [0, T].$$

(b) The closed subset K enjoys the BSVP for the equation (2.1) if and only if:
 $\forall t \in [0, T], \forall \xi \in L^2(\Omega, \mathcal{F}_t, P, K)$, there exists a solution $(Y, Z, U) \in \mathcal{B}_{[0, t]}^2$ to BSDE (2.1) over the time interval $[0, t]$,

$$Y_s = \xi + \int_s^t f(r, Y_r, Z_r, U_r) dr - \int_t^T Z_r dW_r - \int_t^T \int_E U_r(e) \tilde{N}(dr de), s \in [0, t],$$

such that $\{Y_s, s \in [0, t]\}$ is viable in K .

The question we are interested in is that when K enjoys the BSVP for BSDE (2.1).

Let us define for any closed set $K \subset R^m$ the multivalued projection of a point a onto K :

$$\Pi_K(a) := \{b \in K \mid |a - b| = \min_{c \in K} |a - c| = d_K(a)\}.$$

Recall that $\Pi_K(a)$ is a singleton whenever d_K is differentiable at the point a . According to Motzkin's Theorem Π_K is single-valued if and only if K is convex. Notice that $d_K^2(\cdot)$ is convex when K is convex, and thus, due to Alexandrov's Theorem [1], $d_K^2(\cdot)$ is almost everywhere twice differentiable. By twice differentiable, we mean that the function admits a second order Taylor expansion. And this may hold true even if the first derivative is not continuous.

We need an auxiliary result on BSDEs with jumps. Give any $\xi \in L^2(\Omega, \mathcal{F}_T, P, R^m)$, we denote by (Y, Z, U) the unique solution to BSDE (2.1) and by R, V the processes associated to ξ by Lemma 2.2. With these notations we can state

Proposition 2.4. *Suppose that (H1)-(H3) hold true. Then there exist real positive constants C_0, M such that for all $\xi \in L^2(\Omega, \mathcal{F}_T, P, R^m)$,*

$$\begin{aligned} & E[\sup_{s \in [t, T]} |Y_s|^2 | \mathcal{F}_t] + E[\int_t^T |Z_s|^2 ds | \mathcal{F}_t] + E[\int_t^T \int_E |U_s(e)|^2 n(de) ds | \mathcal{F}_t] \\ & \leq C_0 \{E[|\xi|^2 | \mathcal{F}_t] + E[(\int_t^T |f(s, 0, 0, 0)| ds)^2 | \mathcal{F}_t]\}, t \in [0, T], \end{aligned} \tag{2.3}$$

and for all $\varepsilon \in [0, T]$,

$$\begin{aligned} & E[\sup_{s \in [T-\varepsilon, T]} |Y_s - E(\xi | \mathcal{F}_s)|^2 | \mathcal{F}_{T-\varepsilon}] + E[\int_{T-\varepsilon}^T |Z_s - R_s|^2 ds | \mathcal{F}_{T-\varepsilon}] \\ & + E[\int_{T-\varepsilon}^T \int_E |U_s(e) - V_s(e)|^2 n(de) ds | \mathcal{F}_{T-\varepsilon}] \\ & \leq C_0 \varepsilon E[\int_{T-\varepsilon}^T |f(s, E(\xi | \mathcal{F}_s), R_s, V_s)|^2 ds | \mathcal{F}_{T-\varepsilon}], \end{aligned} \tag{2.4}$$

in particular, there exists a positive constant M such that

$$\begin{aligned} & E[|Y_t - E(\xi|\mathcal{F}_t)|^2] + E\left[\int_t^T |Z_s - R_s|^2 ds\right] + E\left[\int_t^T \int_E |U_s(e) - V_s(e)|^2 n(de) ds\right] \\ & \leq (T - t)M, t \in [0, T]. \end{aligned} \quad (2.5)$$

Now we can get the following theorem.

Theorem 2.5. Suppose that $f : \Omega \times [0, T] \times R^m \times R^{m \times d} \times L^2(E, \mathcal{B}_E, n; R^m) \rightarrow R^m$ is a measurable function which satisfies (H1)-(H3). Let K be a nonempty closed set. If K enjoys the BSVP for BSDE (2.1), then the set K is convex.

Proof: The method is totally the same as that in Theorem 2.4 in [7] thanks to (2.5) in Proposition 2.4. So we omit it.

Since the previous theorem means that only convex sets could have the BSVP even with jumps, we restrict our attention to closed convex sets.

Theorem 2.6. Suppose that $f : \Omega \times [0, T] \times R^m \times R^{m \times d} \times L^2(E, \mathcal{B}_E, n; R^m) \rightarrow R^m$ is a measurable function which satisfies (H1)-(H3). Let K be a nonempty closed set. The set K enjoys the BSVP for BSDE (2.1) if and only if:

$\forall (t, z, u) \in [0, T] \times R^{m \times d} \times L^2(E, \mathcal{B}_E, n; R^m)$ and for all $y \in R^m$ such that $d_K^2(\cdot)$ is twice differentiable at y ,

$$\begin{aligned} & 4\langle y - \Pi_K(y), f(t, y, z, u) \rangle \\ & \leq \langle D^2 d_K^2(y) z, z \rangle + C^* d_K^2(y) \\ & \quad + 2 \int_E [d_K^2(y + u(e)) - d_K^2(y) - 2\langle y - \Pi_K(y), u(e) \rangle] n(de), P - a.s., \end{aligned} \quad (2.6)$$

where $C^* > 0$ is a constant which does not depend on (t, y, z, u) .

Let us notice that, under assumption (H2), condition (2.6) takes form:

$$\begin{aligned} & 4\langle y - \Pi_K(y), f(t, \Pi_K(y), z, u) \rangle \\ & \leq \langle D^2 d_K^2(y) z, z \rangle + (C^* + 4L) d_K^2(y) \\ & \quad + 2 \int_E [d_K^2(y + u(e)) - d_K^2(y) - 2\langle y - \Pi_K(y), u(e) \rangle] n(de), P - a.s.. \end{aligned}$$

On the other hand, for some $C' > 0$, the condition

$$\begin{aligned} & 4\langle y - \Pi_K(y), f(t, \Pi_K(y), z, u) \rangle \\ & \leq \langle D^2 d_K^2(y) z, z \rangle + C' d_K^2(y) + \\ & \quad 2 \int_E [d_K^2(y + u(e)) - d_K^2(y) - 2\langle y - \Pi_K(y), u(e) \rangle] n(de), P - a.s. \end{aligned} \quad (2.7)$$

implies (2.6) with constant $C' + 4L$ instead of C^* .

This shows that (2.6) is a condition only on the values of $f(t, \cdot, z, u)$ on ∂K . Recall also that the behavior of d_K on R^m is completely determined by that of ∂K .

Remark 2.7. *Because $D^2[d_K^2]$ is almost everywhere positive semidefinite, and*

$$\begin{aligned} & d_K^2(y + u(e)) - d_K^2(y) - 2\langle y - \Pi_K(y), u(e) \rangle \\ = & (y + u(e) - \Pi_K(y) + \Pi_K(y) - \Pi_K(y + u(e)))^2 - (y - \Pi_K(y))^2 - 2\langle y - \Pi_K(y), u(e) \rangle \\ = & |u(e) + \Pi_K(y) - \Pi_K(y + u(e))|^2 + 2\langle y - \Pi_K(y), \Pi_K(y) - \Pi_K(y + u(e)) \rangle \\ \geq & 0. \end{aligned}$$

The last inequality is from the convexity of K . So a sufficient condition for the BSVP of K with jumps is that there exists some constant $C^* > 0$, such that, for all (t, y, z, u) ,

$$4\langle y - \Pi_K(y), f(t, y, z, u) \rangle \leq C^* d_K^2(y), P - a.s.. \quad (2.7')$$

Example 2.8. *If we set $K = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1\}$. In this case,*

$$d_K^2(x) = \begin{cases} 0, & \text{when } x \in K, \\ (|x| - 1)^2, & \text{when } x \in R^2 \setminus K. \end{cases}$$

Moreover, $d_K^2(\cdot) \in C^1(R^2)$ and twice differentiable when $x \in R^2 \setminus \partial K$. To satisfy (2.7'), we can choose

$$f(t, y, z, u) = y - \Pi_K(y).$$

So obviously, (2.7') holds true with $C^* \geq 4$. In fact the following BSDE with jump does enjoy BSVP for K :

$$Y_t = \xi + \int_t^T (Y_s - \Pi_K(Y_s)) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \tilde{N}(dsde), 0 \leq t \leq T.$$

In deed, the following BSDE with jump has the same solution with above when $\xi \in K$:

$$Y_t = \xi - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \tilde{N}(dsde), 0 \leq t \leq T.$$

So

$$|Y_t|^2 = |E[\xi | \mathcal{F}_t]|^2 \leq E[|\xi|^2 | \mathcal{F}_t] \leq 1.$$

3 The comparison theorem of multidimensional BSDEs with jumps

In this section, with the BSVP theory with jumps we have got in the previous section, we can use the same method as in [10] to study the comparison theorem for multidimensional BSDEs with jumps.

Consider the following BSDEs: $i = 1, 2$,

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i, U_s^i) ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E U_s^i \tilde{N}(dsde), \quad (3.1)$$

where f^1, f^2 satisfy (H1)-(H3), and $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}, P)$. In this subsection, we study the following problem: under which condition the comparison theorem holds? Interestingly, this problem is transformed to a viability problem in $R_+^m \times R^m$ of $(Y^1 - Y^2, Y^2)$.

Theorem 3.1. *Suppose that f^1 and f^2 satisfy (H1)-(H3). Then the following are equivalent:*

(i) *For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; R^m)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0,s]}^2$ to the BSDE (3.1) over interval $[0, s]$ satisfy:*

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) $\forall t \in [0, T], \forall (y, y'), \forall (z, z'), \forall (u, u'),$

$$\begin{aligned} & -4\langle y^-, f^1(t, y^+ + y', z, u) - f^2(t, y', z', u') \rangle \\ \leq & 2 \sum_{k=1}^m I_{\{y_k < 0\}} |z_k - z'_k|^2 + C|y^-|^2 + 2 \sum_{k=1}^m I_{\{y_k \geq 0\}} \int_E |(y_k + u_k(e) - u'_k(e))^-|^2 n(de) \\ & + 2 \sum_{k=1}^m I_{\{y_k < 0\}} \int_E [| (y_k + u_k(e) - u'_k(e))^-|^2 - |y_k^-|^2 - 2y_k(u_k(e) - u'_k(e))] n(de), P - a.s., \end{aligned} \quad (3.2)$$

where C is a constant which dose not depend on $t, (y, y'), (z, z'), (u, u')$.

Proof: Set

$$\bar{Y}_t = (Y_t^1 - Y_t^2, Y_t^2), \bar{Z}_t = (Z_t^1 - Z_t^2, Z_t^2), \bar{U}_t = (U_t^1 - U_t^2, U_t^2).$$

Then (i) is equivalent to the following:

(iii) For any $s \in [0, T], \forall \bar{\xi} = (\bar{\xi}^1, \bar{\xi}^2)$ such that $\bar{\xi}^1 \geq 0$, the unique solution $(\bar{Y}, \bar{Z}, \bar{U})$ to the following BSDE over time interval $[0, s]$:

$$\bar{Y}_t = \bar{\xi} + \int_t^s \bar{f}(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds - \int_t^s \bar{Z}_s dW_s - \int_t^s \int_E \bar{U}_s \tilde{N}(dsde), \quad (3.3)$$

satisfies $\bar{Y}^1 \geq 0$, where for $\bar{y} = (\bar{y}^1, \bar{y}^2)$, $\bar{z} = (\bar{z}^1, \bar{z}^2)$, $\bar{u} = (\bar{u}^1, \bar{u}^2)$,

$$\bar{f}(t, \bar{y}, \bar{z}, \bar{u}) = (f^1(t, \bar{y}^1 + \bar{y}^2, \bar{z}^1 + \bar{z}^2, \bar{u}^1 + \bar{u}^2) - f^2(t, \bar{y}^2, \bar{z}^2, \bar{u}^2), f^2(t, \bar{y}^2, \bar{z}^2, \bar{u}^2)).$$

So we can apply Theorem 2.6 to BSDE (3.3) and the convex closed set $K := R_+^m \times R^m$. Obviously, when $\hat{y} = (y, y') \in K$, (2.7) and (3.2) holds true naturally. When $\hat{y} = (y, y') \in R^{2m} \setminus K$, since $\forall x = (x^1, x^2) \in R^{2m}$,

$$\Pi_K(x) = \begin{pmatrix} (x^1)^+ \\ x^2 \end{pmatrix}, x - \Pi_K(x) = \begin{pmatrix} -(x^1)^- \\ 0 \end{pmatrix}.$$

So

$$d_K^2(x) = |(x^1)^-|^2 = \sum_{k=1}^m I_{\{x_k^1 < 0\}} |x_k^1|^2.$$

And

$$(D^2 d_K^2)(x) \begin{cases} = 0_{2m \times 2m}, & \text{when } x \in K^\circ, \\ \text{does not exist}, & \text{when } x \in \partial K, \\ = (a_{ij})_{2m \times 2m}, & \text{when } x \in R^{2m} \setminus K, \end{cases}$$

where

$$a_{ij} = 0, \quad \text{when } i \neq j, \quad a_{ii} = \begin{cases} 0, & m < i \leq 2m, \\ 0, & 1 \leq i \leq m, x_i^1 \geq 0, \\ 2, & 1 \leq i \leq m, x_i^1 < 0. \end{cases}$$

We can get that (2.7) is equivalent to

$$\begin{aligned} & -4 \langle y^-, f^1(t, y^+ + y', z, u) - f^2(t, y', z', u') \rangle \\ \leq & 2 \sum_{k=1}^m I_{\{y_k < 0\}} |z_k - z'_k|^2 + C |y^-|^2 + 2 \sum_{k=1}^m I_{y_k \geq 0} \int_E |(y_k + u_k(e) - u'_k(e))^-|^2 n(de) \\ & + 2 \sum_{k=1}^m I_{\{y_k < 0\}} \int_E [| (y_k + u_k(e) - u'_k(e))^-|^2 - |y_k|^2 - 2y_k(u_k(e) - u'_k(e))] n(de). \end{aligned}$$

□

Theorem 3.2. *Let $m = 1$. Suppose that f^1 and f^2 satisfy (H1)-(H3). Then the following are equivalent:*

(i) *For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; R)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0, s]}^2$ to the BSDE (3.1) over interval $[0, s]$ satisfy:*

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) $\forall (t, y', z) \in [0, T] \times R \times R^{1 \times d}$,

$$\begin{aligned} & \forall u, u' \in L^2(E, \mathcal{B}_E, n; R) \text{ such that } u \geq u', n(de) - a.s., \\ & f^1(t, y', z, u) - f^2(t, y', z, u') \geq - \int_E (u(e) - u'(e)) n(de). \end{aligned} \tag{3.4}$$

Proof: (i) \Rightarrow (ii). Since $m = 1$, choose $y < 0, z = z', u \geq u'$, from (3.2), we have

$$\begin{aligned} & 4y(f^1(t, y', z', u) - f^2(t, y', z', u')) \\ & \leq Cy^2 + 2 \int_E [| (y + u(e) - u'(e))^- |^2 - y^2 - 2y(u(e) - u'(e))] n(de) \\ & \leq Cy^2 + 2 \int_E -2y(u(e) - u'(e)) n(de). \end{aligned}$$

Dividing by $4y$ and letting y tend to 0, we get (3.4).

(ii) \Rightarrow (i). When $y \geq 0$, (3.2) holds true naturally. When $y < 0$, from (3.4) we have: $\forall t \in [0, T], y', z, z', u, u'$,

$$\begin{aligned} & 4y[f^1(t, y', z, u) - f^2(t, y', z', u')] \\ = & 4y[f^1(t, y', z, u) - f^1(t, y', z', u)] + 4y[f^1(t, y', z', u) - f^2(t, y', z', I_{\{u > u'\}}u' + I_{\{u \leq u'\}}u)] \\ & + 4y[f^2(t, y', z', I_{\{u > u'\}}u' + I_{\{u \leq u'\}}u) - f^2(t, y', z', u')] \\ \leq & 2L^2y^2 + \frac{2}{L^2}|f^1(t, y', z, u) - f^1(t, y', z', u)|^2 + \int_{\{u > u'\}} -4y(u(e) - u'(e))n(de) \\ & + 2L^2y^2 + \frac{2}{L^2}|f^2(t, y', z', I_{\{u > u'\}}u' + I_{\{u \leq u'\}}u) - f^2(t, y', z', u')|^2 \\ \leq & (4L^2 + 2n(E))y^2 + 2|z - z'|^2 + 2 \int_{\{u \leq u'\}} |u(e) - u'(e)|^2 n(de) \\ & + \int_{\{u > u'\}} (-2y^2)n(de) + \int_{\{u > u'\}} -4y(u(e) - u'(e))n(de) \\ \leq & (4L^2 + 2n(E))y^2 + 2|z - z'|^2 + 2 \int_{\{u \leq u'\}} |u(e) - u'(e)|^2 n(de) \\ & + 2 \int_{\{u > u'\}} [| (y + u(e) - u'(e))^- |^2 - y^2 - 2y(u(e) - u'(e))] n(de) \\ = & (4L^2 + n(E))y^2 + 2|z - z'|^2 + 2 \int_E [| (y + u(e) - u'(e))^- |^2 - y^2 - 2y(u(e) - u'(e))] n(de) \end{aligned}$$

□

Remark 3.3. When $m = 1$, (3.4) constrains the form of the dependence of the generators in the jump components of the BSDEs. We refer the reader to Wu [15], Royer [14] for the analogue of (3.4) which can be somewhat contained by (3.4). But the most important of all, (3.4) is a necessary and sufficient condition for the comparison theorem to hold. While in almost any other references on the comparison theorem for 1-dimensional BSDEs with jumps, the results are just sufficient.

Remark 3.4. The following example and counter-example show that (3.4) is really a necessary and sufficient condition for comparison theorem.

Let $E = R \setminus \{0\}$, $n(de) = \delta_1(de)$. Then

$$N_t = \int_0^t \int_E N(dsde), 0 \leq t \leq T,$$

is a standard Poisson process.

(a) If we choose

$$f^1(t, y, z, u) = f^2(t, y, z, u) = -\frac{1}{2}u(1),$$

then $\forall (t, y, z) \in [0, T] \times R \times R^{1 \times d}$ and $\forall u, u' \in L^2(E, \mathcal{B}_E, n; R)$ such that

$$u \geq u', n(de) - a.s.,$$

we have

$$f(t, y, z, u) - f(t, y, z, u') = -\frac{1}{2}(u(1) - u'(1)) \geq -(u(1) - u'(1)) = -\int_E (u(e) - u'(e))n(de).$$

This means that (3.4) holds true. So We know that comparison theorem holds. For example, for any $s \in [0, T]$, if we choose

$$\xi^1 = N_s \geq \xi^2 = 0,$$

then for all $t \in [0, s]$,

$$\begin{aligned} (Y_t^1, Z_t^1, U_t^1) &= (N_t + \frac{1}{2}(s - t), 0, I_{\{e=1\}}), \\ (Y_t^2, Z_t^2, U_t^2) &= (0, 0, 0). \end{aligned}$$

It's clear that $P\{Y_t^1 \geq Y_t^2\} = 1$, for all $0 \leq t \leq s$.

(b) On the other hand, if we choose

$$f^1(t, y, z, u) = f^2(t, y, z, u) = -2u(1)$$

and

$$\xi^1 = N_s, \xi^2 = 0, \forall s \in [0, T],$$

then for all $t \in [0, s]$,

$$\begin{aligned} (Y_t^1, Z_t^1, U_t^1) &= (N_t - (s - t), 0, I_{\{e=1\}}), \\ (Y_t^2, Z_t^2, U_t^2) &= (0, 0, 0). \end{aligned}$$

Obviously, $N_s \geq 0, P - a.s.$, but $P\{Y_t^1 < Y_t^2\} > 0$, for all $0 \leq t \leq s$. This is because, in this case,

$$f(t, y, z, u) - f(t, y, z, u') = -2(u(1) - u'(1)) < -(u(1) - u'(1)) = -\int_E (u(e) - u'(e))n(de).$$

Corollary 3.5. *Let $m = 1$ and suppose furthermore that f^1 and f^2 are independent of u and satisfy (H1)-(H3). Then the following are equivalent:*

(i) *For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; R)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0,s]}^2$ to the BSDE (3.1) over interval $[0, s]$ satisfy:*

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) $f^1(t, y, z) \geq f^2(t, y, z), \forall t \in [0, T], \forall (y, z) \in R \times R^{1 \times d}$.

This generalizes the result in [10].

Now consider the special case when $f^1 = f^2 = f$. Then we have:

Theorem 3.6. *Suppose that f satisfies (H1)-(H3). Then the following are equivalent:*

(i) *For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; R^m)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0,s]}^2$ to the BSDE (3.1) over interval $[0, s]$ satisfy:*

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) *For any $k = 1, 2, \dots, m$,*

$$\left\{ \begin{array}{l} \text{(a)} \quad f_k \text{ depends only on } z_k; \\ \text{(b)} \quad f_k(t, \delta^k y + y', z_k, u) - f_k(t, y', z_k, u') \geq - \int_E (u_k(e) - u'_k(e)) n(de), \\ \quad \forall u \geq u', n(de) - a.s., \text{ for any } \delta^k y \in R^m \text{ such that } \delta^k y \geq 0, (\delta^k y)_k = 0; \\ \text{(c)} \quad \text{There exists a positive constant } C_0, \text{ s.t.,} \\ \quad \forall y, y', z, u, u' \text{ satisfy } u \leq u', n(de) - a.s., \\ \quad \sum_{k=1}^m I_{\{y_k < 0\}} 4y_k (f_k(t, y^+ + y', z_k, u) - f_k(t, y', z_k, u')) \\ \quad \leq C_0 |y^-|^2 + 2 \sum_{k=1}^m I_{\{y_k < 0\}} \int_E |u_k(e) - u'_k(e)|^2 n(de) \\ \quad + 2 \sum_{k=1}^m I_{\{y_k \geq 0\}} \int_E |(y_k + u_k(e) - u'_k(e))^-|^2 n(de). \end{array} \right. \quad (3.5)$$

Proof: From Theorem 3.1, (i) is equivalent to (3.2) with $f^1 = f^2$. What we have to do is to prove that, when $f^1 = f^2$, (3.2) \Leftrightarrow (ii).

Let us pick $y_k < 0$, and $y = y_k e_k, z_k = z'_k, u = u'$, from (3.2), we have

$$4y_k (f_k(t, y', z, u) - f_k(t, y', z', u)) \leq C y_k^2.$$

We deduce easily that

$$|f_k(t, y', z, u) - f_k(t, y', z', u)| = 0, \quad \text{when } z_k = z'_k.$$

So obviously, f_k depends only on z_k .

Moreover, for $\delta^k y \in R^m$ such that $\delta^k y \geq 0$, $(\delta^k y)_k = 0$, putting in (3.2) with $y = \delta^k y - \varepsilon e_k$, $\varepsilon > 0$, $z = z'$, $u(e) \geq u'(e)$, $n(de) - a.s.$, we have

$$\begin{aligned} & -4\varepsilon(f_k(t, \delta^k y + y', z_k, u) - f_k(t, y', z_k, u')) \\ & \leq C\varepsilon^2 + 2 \int_E [|(-\varepsilon + u_k(e) - u'_k(e))^-|^2 - \varepsilon^2 + 2\varepsilon(u_k(e) - u'_k(e))] n(de). \end{aligned}$$

Dividing by -4ε and letting $\varepsilon \rightarrow 0$, we hence get (b) in (3.5).

For (c), it's straightforward by (3.2) with $u \leq u'$, $n(de) - a.s.$.

(ii) \Rightarrow (3.2). When $y \geq 0$, (3.2) holds true naturally. When there exist $1 \leq k \leq n$, such that, $y_k < 0$. $\forall t \in [0, T]$, y', z, z', u, u' , we set $\hat{u} = (\hat{u}_k)_{k=1}^m$, where

$$\hat{u}_k = I_{\{u_k > u'_k\}} u'_k + I_{\{u_k \leq u'_k\}} u_k.$$

So we have $u \geq \hat{u}$, $\hat{u} \leq u'$. Then from (ii) we have:

$$\begin{aligned} & \sum_{k=1}^m I_{\{y_k < 0\}} 4y_k [f_k(t, y^+ + y', z_k, u) - f_k(t, y', z'_k, u')] \\ = & \sum_{k=1}^m I_{\{y_k < 0\}} 4y_k [f_k(t, y^+ + y', z_k, u) - f_k(t, y^+ + y', z'_k, u) \\ & \quad + f_k(t, y^+ + y', z'_k, u) - f_k(t, y^+ + y', z'_k, \hat{u}) \\ & \quad + f_k(t, y^+ + y', z'_k, \hat{u}) - f_k(t, y', z'_k, u')] \\ \leq & \sum_{k=1}^m I_{\{y_k < 0\}} [(2L^2 + 2n(E) + C_0)y_k^2 + 2|z_k - z'_k|^2 + 2 \int_{\{u_k \leq u'_k\}} |u_k(e) - u'_k(e)|^2 n(de)] \\ & + \sum_{k=1}^m I_{\{y_k < 0\}} \int_{\{u_k > u'_k\}} 2[|(y_k + u_k(e) - u'_k(e))^-|^2 - y_k^2 - 2y_k(u_k(e) - u'_k(e))n(de)] \\ & + 2 \sum_{k=1}^m I_{\{y_k \geq 0\}} \int_E |(y_k + u_k(e) - u'_k(e))^-|^2 n(de). \end{aligned}$$

Remark 3.7. When f_k depends only on u_k , with (b) in (3.5) and the Lipschitz condition of f w.r.t. u we have

$$\begin{aligned} & \sum_{k=1}^m I_{\{y_k < 0\}} 4y_k (f_k(t, y^+ + y', z_k, u_k) - f_k(t, y', z_k, u'_k)) \\ = & \sum_{k=1}^m I_{\{y_k < 0\}} 4y_k (f_k(t, y^+ + y', z_k, u_k) - f_k(t, y', z_k, u_k) + f_k(t, y', z_k, u_k) - f_k(t, y', z_k, u'_k)) \\ \leq & \sum_{k=1}^m I_{\{y_k < 0\}} 4y_k (f_k(t, y', z_k, u_k) - f_k(t, y', z_k, u'_k)) \\ \leq & 2L^2 |y^-|^2 + 2 \sum_{k=1}^m I_{\{y_k < 0\}} \int_E |u_k(e) - u'_k(e)|^2 n(de). \end{aligned}$$

So the condition (c) in (3.5) can be cancelled:

Theorem 3.8. Suppose that f satisfies (H1)-(H3) and f_k depends only on u_k . Then the following are equivalent:

(i) For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; R^m)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0,s]}^2$ to the BSDE (3.1) over interval $[0, s]$ satisfy:

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) For any $k = 1, 2, \dots, m$,

$$\left\{ \begin{array}{l} \text{(a)} \quad f_k \text{ depends only on } z_k; \\ \text{(b)} \quad \forall u \geq u', n(de) - a.s., \text{ for any } \delta^k y \in R^m \text{ such that } \delta^k y \geq 0, (\delta^k y)_k = 0, \\ \quad f_k(t, \delta^k y + y', z_k, u_k) - f_k(t, y', z_k, u'_k) \geq - \int_E (u_k(e) - u'_k(e)) n(de). \end{array} \right.$$

Corollary 3.9. If we suppose that f is independent of u , then the following are equivalent:

(i) For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; R^m)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0,s]}^2$ to the BSDE (3.1) over interval $[0, s]$ satisfy:

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) For any $k = 1, 2, \dots, n$,

$$f_k \text{ depends only on } z_k \text{ and}$$

$$f_k(t, \delta^k y + y', z_k) \geq f_k(t, y', z_k), \text{ for any } \delta^k y \in R^m \text{ such that } \delta^k y \geq 0, (\delta^k y)_k = 0.$$

This generalizes the result in [10].

At the end of this section, we study the comparison theorem for multidimensional BSDEs with jumps.

We define \mathbb{S}^m as the space of symmetric real $m \times m$ matrices, and denote by \mathbb{S}_+^m the subspace of \mathbb{S}^m containing the nonnegative elements in \mathbb{S}^m . Without loss of generality, we set $d = 1$.

Let us again consider a function F , which will be in the sequel the generator of the BSDE, defined on $\Omega \times [0, T] \times \mathbb{S}^m \times \mathbb{S}^m \times \mathbb{S}^m$, with values in \mathbb{S}^m , such that the process $(F(t, y, z, u))_{t \in [0, T]}$ is a progressively measurable process for each $(y, z, u) \in \mathbb{S}^m \times \mathbb{S}^m \times \mathbb{S}^m$.

We consider the following matrix-valued BSDE with jump:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \tilde{N}(ds de), \quad (3.6)$$

where $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{S}^m)$.

We want to study when the comparison theorem holds for two BSDE of type (3.6).

Consider the following two BSDEs: $i = 1, 2$,

$$Y_t^i = \xi^i + \int_t^T F^i(s, Y_s^i, Z_s^i, U_s^i) ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E U_s^i(e) \tilde{N}(ds de), \quad (3.7)$$

where F^1, F^2 satisfy (H1)-(H3), and $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}, P; \mathbb{S}^m)$. We study the same problem as that in the preceding content: under which condition the comparison theorem holds for matrix-valued BSDEs? Interestingly, this problem is transformed again to a viability problem in $\mathbb{S}_+^m \times \mathbb{S}^m$.

Theorem 3.10. *Suppose that F^1 and F^2 satisfy (H1)-(H3). Then the following are equivalent:*

(i) *For any $s \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_s, P; \mathbb{S}^m)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) in $\mathcal{B}_{[0,s]}^2$ valued in $\mathbb{S}^m \times \mathbb{S}^m \times \mathbb{S}^m$ to the BSDE (3.7) over time interval $[0, s]$ satisfy:*

$$Y_t^1 \geq Y_t^2, \quad t \in [0, s];$$

(ii) $\forall t \in [0, T], \forall (y, y'), \forall (z, z'), \forall (u, u'),$

$$\begin{aligned} & -4\langle y^-, F^1(t, y^+ + y', z, u) - F^2(t, y', z', u') \rangle \\ \leq & \langle D^2 d_{\mathbb{S}_+^m}^2(y)(z - z'), z - z' \rangle + C\|y^-\|^2 \\ & + 2 \int_E [\|(y + u(e) - u'(e))^- \|^2 - \|y^-\|^2 + 2\langle y^-, u(e) - u'(e) \rangle] n(de), P - a.s., \end{aligned}$$

where C is a constant which dose not depend on $t, (y, y'), (z, z'), (u, u')$.

Proof: We can see from the appendix in [10] that, for any $y \in \mathbb{S}^m$, y has an expression:

$$y(\lambda, A) = e^A \sum_{i=1}^m \lambda_i e_i e_i^T e^{-A},$$

where A is an antisymmetric real $m \times m$ matrix ($A^T = -A$), $\lambda_i \in R$, $\{e_1, e_2, \dots, e_m\}$ is the standard basis of R^m .

If we set

$$y^+(\lambda, A) = e^A \sum_{i=1}^m \lambda_i^+ e_i e_i^T e^{-A}, \quad y^-(\lambda, A) = e^A \sum_{i=1}^m \lambda_i^- e_i e_i^T e^{-A}.$$

Then from [10], we have

$$d_{\mathbb{S}_+^m}^2(y) = \|y^-\|^2, \quad \Pi_{\mathbb{S}_+^m}(y) = y^+, \quad \text{and} \quad \nabla d_{\mathbb{S}_+^m}^2(y) = -2y^-,$$

where $\|y\| = (tr(y^2))^{\frac{1}{2}}$. So we can use the same method with that in Theorem 3.1 to finish the proof of this theorem. \square

Corollary 3.11. *Let us suppose furthermore that F^1 and F^2 are independent of z and u . Then (ii) is equivalent to:*

$$(ii') - 4\langle y^-, F^1(t, y^+ + y') - F^2(t, y') \rangle \leq C\|y^-\|^2.$$

This generalizes the result in [10].

4 Appendix: Proofs of main theorem

Proof of Proposition 2.4. Applying Itô's formula to $e^{\beta t}|Y_t|^2$, we have for $s \in [t, T]$,

$$\begin{aligned} & e^{\beta s}|Y_s|^2 + \int_s^T e^{\beta r}(\beta|Y_r|^2 + |Z_r|^2)dr + \int_s^T \int_E e^{\beta r}|U_r(e)|^2 n(de)dr \\ &= e^{\beta T}|\xi|^2 + 2 \int_s^T e^{\beta r} \langle f(r, Y_r, Z_r, U_r), Y_r \rangle dr - 2 \int_s^T e^{\beta r} \langle Y_r, Z_r dW_r \rangle \\ & \quad - \int_s^T \int_E e^{\beta r}(|Y_{r-} + U_r(e)|^2 - |Y_{r-}|^2) \tilde{N}(drde). \end{aligned}$$

Since

$$\begin{aligned} 2\langle f(r, y, z, u), y \rangle &= 2\langle f(r, y, z, u) - f(r, 0, 0, 0), y \rangle + 2\langle f(r, 0, 0, 0), y \rangle \\ &\leq \frac{1}{2}|z|^2 + \frac{1}{2}\|u\|^2 + 2(L + 2L^2)|y|^2 + 2\langle f(r, 0, 0, 0), y \rangle, \end{aligned}$$

and then, for $s \in [t, T]$,

$$\begin{aligned} & e^{\beta s}|Y_s|^2 + \int_s^T e^{\beta r}[(\beta - 2L - 4L^2)|Y_r|^2 + \frac{1}{2}|Z_r|^2]dr + \int_s^T \int_E \frac{1}{2}e^{\beta r}|U_r(e)|^2 n(de)dr \\ &\leq e^{\beta T}|\xi|^2 + 2 \int_s^T e^{\beta r} \langle f(r, 0, 0, 0), Y_r \rangle dr - 2 \int_s^T e^{\beta r} \langle Y_r, Z_r dW_r \rangle \\ & \quad - \int_s^T \int_E e^{\beta r}(|Y_{r-} + U_r(e)|^2 - |Y_{r-}|^2) \tilde{N}(drde). \end{aligned} \tag{4.1}$$

By replacing $\beta = 2L + 4L^2$, inequality (4.1) yields

$$\begin{aligned} & \int_s^T e^{\beta r}|Z_r|^2 dr + \int_s^T \int_E e^{\beta r}|U_r(e)|^2 n(de)dr \\ &\leq 2e^{\beta T}|\xi|^2 + 4 \int_s^T e^{\beta r} \langle f(r, 0, 0, 0), Y_r \rangle dr - 4 \int_s^T e^{\beta r} \langle Y_r, Z_r dW_r \rangle \\ & \quad - 2 \int_s^T \int_E e^{\beta r}(|Y_{r-} + U_r(e)|^2 - |Y_{r-}|^2) \tilde{N}(drde). \end{aligned} \tag{4.2}$$

By the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
& E[\sup_{s \in [t, T]} |\int_s^T e^{\beta r} \langle Y_r, Z_r dW_r \rangle| | \mathcal{F}_t] \\
& \leq 2E[\sup_{s \in [t, T]} |\int_t^s e^{\beta r} \langle Y_r, Z_r dW_r \rangle| | \mathcal{F}_t] \\
& \leq CE[(\int_t^T e^{2\beta r} |Y_r|^2 |Z_r|^2 dr)^{\frac{1}{2}} | \mathcal{F}_t] \\
& \leq CE\{[\sup_{s \in [t, T]} (e^{\frac{1}{2}\beta s} |Y_s|)(\int_t^T e^{\beta r} |Z_r|^2 dr)^{\frac{1}{2}}] | \mathcal{F}_t\} \\
& \leq \frac{1}{8}E[\sup_{s \in [t, T]} e^{\beta s} |Y_s|^2 | \mathcal{F}_t] + CE[\int_t^T e^{\beta r} |Z_r|^2 dr | \mathcal{F}_t],
\end{aligned} \tag{4.3}$$

here and in the sequel, C denote some positive constant depending only on L and T , and to which we allow to change from one formula to the other. And

$$\begin{aligned}
& E[\sup_{s \in [t, T]} \int_s^T \int_E e^{\beta r} (|Y_{r-} + U_r(e)|^2 - |Y_{r-}|^2) \tilde{N}(dr de) | \mathcal{F}_t] \\
& \leq E[\int_t^T \int_E e^{\beta r} |U_r(e)|^2 N(dr de) + \int_t^T \int_E e^{\beta r} |U_r(e)|^2 n(de) dr | \mathcal{F}_t] \\
& \quad + 2E\{\sup_{s \in [t, T]} [\int_t^s \int_E e^{\beta r} 2\langle Y_{r-}, U_r(e) \rangle \tilde{N}(dr de) | \mathcal{F}_t]\} \\
& \leq 2E[\int_t^T \int_E e^{\beta r} |U_r(e)|^2 n(de) dr | \mathcal{F}_t] + CE[(\int_t^T \int_E e^{2\beta r} |Y_r|^2 |U_r(e)|^2 n(de) dr)^{\frac{1}{2}} | \mathcal{F}_t] \\
& \leq 2E[\int_t^T \int_E e^{\beta r} |U_r(e)|^2 n(de) dr | \mathcal{F}_t] + CE[(\sup_{s \in [t, T]} e^{\frac{1}{2}\beta s} |Y_s|)(\int_t^T \int_E e^{\beta r} |U_r(e)|^2 n(de) dr)^{\frac{1}{2}} | \mathcal{F}_t] \\
& \leq \frac{1}{4}E[\sup_{s \in [t, T]} e^{\beta s} |Y_s|^2 | \mathcal{F}_t] + CE[\int_t^T \int_E e^{\beta r} |U_r|^2 n(de) dr | \mathcal{F}_t]
\end{aligned}$$

This with (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}
& \frac{1}{2}E[\sup_{s \in [t, T]} e^{\beta s} |Y_s|^2 | \mathcal{F}_t] \\
& \leq Ce^{\beta T} E[|\xi|^2 | \mathcal{F}_t] + CE[\int_t^T e^{\beta r} |Y_r| |f(r, 0, 0, 0)| dr | \mathcal{F}_t] \\
& \leq Ce^{\beta T} E[|\xi|^2 | \mathcal{F}_t] + CE[(\sup_{s \in [t, T]} e^{\frac{1}{2}\beta s} |Y_s|) \int_t^T e^{\frac{1}{2}\beta r} |f(r, 0, 0, 0)| dr | \mathcal{F}_t].
\end{aligned}$$

Hence,

$$E[\sup_{s \in [t, T]} e^{\beta s} |Y_s|^2 | \mathcal{F}_t] \leq Ce^{\beta T} E[|\xi|^2 | \mathcal{F}_t] + CE[(\int_t^T e^{\frac{1}{2}\beta r} |f(r, 0, 0, 0)| dr)^2 | \mathcal{F}_t].$$

This, together with (4.2), gives

$$\begin{aligned} & E\left[\int_t^T e^{\beta s} |Z_s|^2 ds + \int_t^T \int_E e^{\beta s} |U_s(e)|^2 n(de) ds \middle| \mathcal{F}_t\right] \\ & \leq C e^{\beta T} E[|\xi|^2 | \mathcal{F}_t] + C E\left[\left(\int_t^T e^{\frac{1}{2}\beta r} |f(r, 0, 0, 0)| dr\right)^2 \middle| \mathcal{F}_t\right]. \end{aligned}$$

So

$$\begin{aligned} & E[\sup_{s \in [t, T]} e^{\beta s} |Y_s|^2 | \mathcal{F}_t] + E\left[\int_t^T e^{\beta s} |Z_s|^2 ds + \int_t^T \int_E e^{\beta s} |U_s(e)|^2 n(de) ds \middle| \mathcal{F}_t\right] \\ & \leq C e^{\beta T} E[|\xi|^2 | \mathcal{F}_t] + C E\left[\left(\int_t^T e^{\frac{1}{2}\beta r} |f(r, 0, 0, 0)| dr\right)^2 \middle| \mathcal{F}_t\right]. \end{aligned}$$

Choose $C_0 = C e^{\beta T}$, one can obtain (2.3).

In order to prove (2.4), note that BSDE (2.1) is equivalent to

$$\bar{Y}_t = \int_t^T g(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_E \bar{U}_s(e) \tilde{N}(ds de), t \in [0, T],$$

where $\bar{Y}_t := Y_t - E(\xi | \mathcal{F}_t)$, $\bar{Z}_t := Z_t - R_t$, $\bar{U}_t := U_t - V_t$ and

$$g(s, \bar{y}, \bar{z}, \bar{u}) := f(s, \bar{y} + E(\xi | \mathcal{F}_s), \bar{z} + R_s, \bar{u} + V_s).$$

Then inequality (2.3) applied to $(\bar{Y}_t, \bar{Z}_t, \bar{U}_t)$ implies (2.4).

To prove (2.5), we set $\varepsilon = T - t$ in (2.4) and take expectation. Then the left thing to us is to prove, there exists a positive constant M_0 such that

$$E\left[\int_t^T |f(s, E(\xi | \mathcal{F}_s), R_s, V_s)|^2 ds\right] \leq M_0. \quad (4.4)$$

This is because

$$\begin{aligned} & E\left[\int_t^T |f(s, E(\xi | \mathcal{F}_s), R_s, V_s)|^2 ds\right] \\ & \leq E\left[\int_t^T 4(|f(s, 0, 0, 0)|^2 + L^2 |E(\xi | \mathcal{F}_s)|^2 + L^2 |R_s|^2 + L^2 |V_s|^2) ds\right] \\ & \leq 4TE[\sup_{s \in [t, T]} |f(s, 0, 0, 0)|^2] + 4TL^2 E|\xi|^2 \\ & \quad + 4L^2 E \int_t^T |R_s|^2 ds + 4L^2 E \int_t^T \int_E |V_s(e)|^2 n(de) ds. \end{aligned}$$

By (H3) and Lemma 2.2, we can deduce that there exists $M_0 > 0$, such that (4.4) holds true. So we complete the proof of the proposition. \square

Proof of Theorem 2.6. This proof is splitted into several steps.

(a) Necessity. Let $t \in (0, T]$ and $\varepsilon > 0$ be such that, for some

$$t_* \geq 0, t_\varepsilon := t - \varepsilon > t_* \geq 0.$$

Fix $y \in R^m$, $z \in R^{m \times d}$, $u \in L^2(E, \mathcal{B}_E, n; R^m)$ and

$$\xi := y + z(W_t - W_{t_\varepsilon}) + \int_{t_\varepsilon}^t \int_E u(e) N(drde).$$

Denote by (Y, Z, U) the unique solution to BSDE with jump

$$Y_s = \xi + \int_s^t f(r, Y_r, Z_r, Z_r) dr - \int_s^t Z_r dW_r - \int_s^t \int_E U_r(e) \tilde{N}(drde), s \in [t_\varepsilon, t].$$

Furthermore, we introduce the Process $\hat{Y} := \{\hat{Y}_s, s \in [t_\varepsilon, t]\}$ as follows:

$$\begin{aligned} \hat{Y}_s &= \xi + (t-s)f(t_\varepsilon, y, z, u) - z(W_t - W_s) - \int_s^t \int_E u(e) \tilde{N}(drde) \\ &= y + (t-s)f(t_\varepsilon, y, z, u) + z(W_s - W_{t_\varepsilon}) + (t-s) \int_E u(e) n(de) + \int_{t_\varepsilon}^s \int_E u(e) N(drde). \end{aligned}$$

Let's compute that

$$\begin{aligned} & E[|\xi|^2 | \mathcal{F}_{t_*}] \\ &= E[y + z(W_t - W_{t_\varepsilon}) + \int_{t_\varepsilon}^t \int_E u(e) \tilde{N}(drde) + \int_{t_\varepsilon}^t \int_E u(e) n(de) dr]^2 \\ &\leq 4E[|y|^2 + |z(W_t - W_{t_\varepsilon})|^2 + |\int_{t_\varepsilon}^t \int_E u(e) \tilde{N}(drde)|^2 + |\int_{t_\varepsilon}^t \int_E u(e) n(de) dr|^2] \\ &= 4[|y|^2 + \varepsilon z^2 + \int_{t_\varepsilon}^t \int_E |u(e)|^2 n(de) dr + |\int_{t_\varepsilon}^t \int_E u(e) n(de) dr|^2] \\ &\leq 4[|y|^2 + \varepsilon z^2 + (\varepsilon + n(E)\varepsilon^2) \int_E |u(e)|^2 de], \end{aligned}$$

and

$$E[(\int_{t_\varepsilon}^t |f(r, 0, 0, 0)| dr)^2 | \mathcal{F}_{t_*}] \leq E[\varepsilon^2 \sup_{s \in [t_\varepsilon, t]} |f(s, 0, 0, 0)|^2 | \mathcal{F}_{t_*}]$$

So according to (2.3) in Proposition 2.4, there exists a nonnegative random variable $\zeta \in L^1(\Omega, \mathcal{F}_{t_*}, P)$ whose norm depends only on y, z and u , such that

$$E[\sup_{s \in [t_\varepsilon, t]} |Y_s|^2 | \mathcal{F}_{t_*}] + E[\int_{t_\varepsilon}^t |Z_s|^2 ds | \mathcal{F}_{t_*}] + E[\int_{t_\varepsilon}^t \int_E |U_s(e)|^2 n(de) ds | \mathcal{F}_{t_*}] \leq \zeta. \quad (4.5)$$

(In the sequel, we will denote by ζ a nonnegative random variable which belongs to $L^1(\Omega, \mathcal{F}_{t_*}, P)$ and whose norm depends only on y, z and u , and we allow it to change from one formula to the other and assume that $\zeta \geq 1, P - a.s.$ if we need.) By (2.4) in Proposition 2.4,

$$\begin{aligned}
& E[\sup_{s \in [t_\varepsilon, t]} |Y_s - E(\xi | \mathcal{F}_s)|^2 | \mathcal{F}_{t_*}] + E[\int_{t_\varepsilon}^t |Z_s - z|^2 ds | \mathcal{F}_{t_*}] \\
& + E[\int_{t_\varepsilon}^t \int_E |U_s(e) - u(e)|^2 n(de) ds | \mathcal{F}_{t_*}] \\
\leq & C_0 \varepsilon E[\int_{t_\varepsilon}^t |f(s, E(\xi | \mathcal{F}_s), z, u)|^2 ds | \mathcal{F}_{t_*}] \\
\leq & C_0 \varepsilon E[\int_{t_\varepsilon}^t 4(|f(s, 0, 0, 0)|^2 + L^2 |E(\xi | \mathcal{F}_s)|^2 + L^2 |z|^2 + L^2 \|u\|^2) ds | \mathcal{F}_{t_*}] \\
\leq & \varepsilon^2 4C_0 E[(\sup_{s \in [t_\varepsilon, t]} |f(s, 0, 0, 0)|^2) + L^2 |\xi|^2 + L^2 |z|^2 + L^2 \|u\|^2 | \mathcal{F}_{t_*}] \\
\leq & \zeta \varepsilon^2.
\end{aligned} \tag{4.6}$$

Then

$$\begin{aligned}
& E[\sup_{s \in [t_\varepsilon, t]} |Y_s - y|^2 | \mathcal{F}_{t_*}] \\
\leq & 4E[\sup_{s \in [t_\varepsilon, t]} |Y_s - E(\xi | \mathcal{F}_s)|^2 | \mathcal{F}_{t_*}] + 4E[\sup_{s \in [t_\varepsilon, t]} |z(W_s - W_{t_\varepsilon})|^2 | \mathcal{F}_{t_*}] \\
& + 4E[(\varepsilon \int_E |u(e)| n(de))^2 | \mathcal{F}_{t_*}] + 4E[\sup_{s \in [t_\varepsilon, t]} |\int_{t_\varepsilon}^s \int_E u(e) \tilde{N}(dr de)|^2 | \mathcal{F}_{t_*}] \\
\leq & 4[\zeta \varepsilon^2 + \varepsilon |z|^2 + (n(E) \varepsilon^2 + C \varepsilon) \|u\|^2] \\
\leq & \zeta \varepsilon, \quad \text{for } \varepsilon \in (0, t - t_*).
\end{aligned} \tag{4.7}$$

Since for $t_\varepsilon \leq s \leq t$,

$$\begin{aligned}
& Y_s - \hat{Y}_s \\
= & \int_s^t (f(r, Y_r, Z_r, U_r) - f(t_\varepsilon, y, z, u)) dr - \int_s^t (Z_r - z) dW_r - \int_s^t \int_E (U_r(e) - u(e)) \tilde{N}(dr de),
\end{aligned}$$

we can apply Itô's formula to $|Y_s - \hat{Y}_s|^2$ over $[t_\varepsilon, t]$ and with the same technique as that in Proposition 2.4, noting (4.6) and (4.7), we have

$$\begin{aligned}
& E[\sup_{s \in [t_\varepsilon, t]} |Y_s - \hat{Y}_s|^2 | \mathcal{F}_{t_*}] + E[\int_{t_\varepsilon}^t |Z_r - z|^2 dr | \mathcal{F}_{t_*}] \\
& + E[\int_{t_\varepsilon}^t \int_E |U_r(e) - u(e)|^2 n(de) dr | \mathcal{F}_{t_*}] \\
\leq & C \varepsilon E[\int_{t_\varepsilon}^t |f(r, Y_r, Z_r, U_r) - f(t_\varepsilon, y, z, u)|^2 dr | \mathcal{F}_{t_*}] \\
\leq & 4C \varepsilon^2 \{E[\sup_{s \in [t_\varepsilon, t]} |f(s, y, z, u) - f(t_\varepsilon, y, z, u)|^2 | \mathcal{F}_{t_*}] + L^2 E[\sup_{s \in [t_\varepsilon, t]} |Y_s - y|^2 | \mathcal{F}_{t_*}]\} \\
& + 4CL^2 \varepsilon \{E[\int_{t_\varepsilon}^t |Z_r - z|^2 dr | \mathcal{F}_{t_*}] + E[\int_{t_\varepsilon}^t \int_E |U_r(e) - u(e)|^2 n(de) dr | \mathcal{F}_{t_*}]\} \\
\leq & \zeta \varepsilon^2 \beta_\varepsilon^1,
\end{aligned} \tag{4.8}$$

where

$$\beta_\varepsilon^1 = 8CL^2\varepsilon + 4\frac{C}{\zeta}E\left[\sup_{s\in[t_\varepsilon, t]} |f(s, y, z, u) - f(t_\varepsilon, y, z, u)|^2 \middle| \mathcal{F}_{t_*}\right].$$

Observe that by (H1):

$$\sup_{s\in[t_\varepsilon, t]} |f(s, y, z, u) - f(t_\varepsilon, y, z, u)|^2 \rightarrow 0, \quad \text{when } \varepsilon \rightarrow 0, P - a.s.,$$

and that

$$\begin{aligned} & \sup_{s\in[t_\varepsilon, t]} |f(s, y, z, u) - f(t_\varepsilon, y, z, u)|^2 \\ & \leq 4 \sup_{t\in[0, T]} |f(t, y, z, u)|^2 \\ & \leq 8 \sup_{t\in[0, T]} |f(t, 0, 0, 0)|^2 + 8L^2(|y|^2 + |z|^2 + \|u\|^2) \in L^1(\Omega, \mathcal{F}_T, P). \end{aligned}$$

Hence from Lebesgue's dominated convergence theorem, $\beta_\varepsilon^1 \rightarrow 0, P - a.s.$, as $\varepsilon \rightarrow 0$.

Let us now establish two auxiliary results on the processes Y and \hat{Y} which will enable us to finish the proof of the necessity.

Lemma 4.1. *Under the assumptions made above, there is some nonnegative random variable $\zeta \in L^1(\Omega, \mathcal{F}_{t_*}, P)$ whose norm depends only on y, z and u such that*

$$E[|d_K^2(Y_{t_\varepsilon}) - d_K^2(\hat{Y}_{t_\varepsilon})| \middle| \mathcal{F}_{t_*}] \leq \zeta \varepsilon \sqrt{\beta_\varepsilon^1},$$

for any $\varepsilon > 0$ with $t - \varepsilon > t_*$.

Proof of Lemma 4.1. Note that, since $K \neq \emptyset$, there exists some constant $C > 0$ such that, $\forall x, x' \in R^m$

$$|d_K^2(x) - d_K^2(x')| = |d_K(x) - d_K(x')|(d_K(x) + d_K(x')) \leq C(1 + |x| + |x'|)|x - x'|.$$

So from (4.8)

$$\begin{aligned} & E[|d_K^2(Y_{t_\varepsilon}) - d_K^2(\hat{Y}_{t_\varepsilon})| \middle| \mathcal{F}_{t_*}] \\ & \leq CE[(1 + |Y_{t_\varepsilon}| + |\hat{Y}_{t_\varepsilon}|)|Y_{t_\varepsilon} - \hat{Y}_{t_\varepsilon}| \middle| \mathcal{F}_{t_*}] \\ & \leq C(E[(1 + |Y_{t_\varepsilon}| + |\hat{Y}_{t_\varepsilon}|)^2 \middle| \mathcal{F}_{t_*}])^{\frac{1}{2}}(E[|Y_{t_\varepsilon} - \hat{Y}_{t_\varepsilon}|^2 \middle| \mathcal{F}_{t_*}])^{\frac{1}{2}} \\ & \leq C\varepsilon \sqrt{\zeta \beta_\varepsilon^1} (E[(1 + |Y_{t_\varepsilon}| + |\hat{Y}_{t_\varepsilon}|)^2 \middle| \mathcal{F}_{t_*}])^{\frac{1}{2}}. \end{aligned}$$

Observe that $\hat{Y}_{t_\varepsilon} = y + \varepsilon f(t_\varepsilon, y, z, u) + \varepsilon \int_E u(e)n(de)$, noting (H3) and (4.5), we have

$$E[|d_K^2(Y_{t_\varepsilon}) - d_K^2(\hat{Y}_{t_\varepsilon})| \middle| \mathcal{F}_{t_*}] \leq \zeta \varepsilon \sqrt{\beta_\varepsilon^1},$$

for some $\zeta \in L^1(\Omega, \mathcal{F}_{t_*}, P)$. This completes the proof of Lemma 4.1.

□

Lemma 4.2. *We can find some \mathcal{F}_{t_*} -measurable random variable $\gamma_\varepsilon = \gamma_\varepsilon(y, z, u)$ with $\liminf_{\varepsilon \rightarrow 0} \gamma_\varepsilon \geq 0$ such that*

$$\begin{aligned} & E[d_K^2(\hat{Y}_{t_\varepsilon}) - d_K^2(\xi)|\mathcal{F}_{t_*}] \\ \geq & \varepsilon \{E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e)n(de) \rangle |\mathcal{F}_{t_*}] \\ & - \frac{1}{2} \langle D^2[d_K^2(y)]z, z \rangle - \frac{1}{\varepsilon} E[\int_{t_\varepsilon}^t \int_E (d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-}))n(de)ds |\mathcal{F}_{t_*}] + \gamma_\varepsilon\}, \end{aligned}$$

where $\xi_s = y + z(W_s - W_{t_\varepsilon}) + \int_{t_\varepsilon}^s \int_E u(e)N(drde)$, $s \in [t_\varepsilon, t]$.

Proof: We know that the function $d_K^2(\cdot)$ is twice differentiable almost everywhere. Let us denote by Λ_K the set of all points of R^m where d_K^2 is twice differentiable. This set is of full Lebesgue measure. Let us fix now $y \in \Lambda_K$ and define the following function $\alpha : R^m \rightarrow R$:

$$\alpha(x) := d_K^2(x + y) - d_K^2(y) - \langle \nabla d_K^2(y), x \rangle - \frac{1}{2} \langle D^2[d_K^2(y)]x, x \rangle.$$

There are two following properties of $\alpha(\cdot)$ (see [7]):

$$\lim_{|x| \rightarrow 0} \frac{\alpha(x)}{|x|^2} = 0,$$

$$\forall x \in R^m, \quad \alpha(x) \leq |x|^2(1 + |D^2 d_K^2(y)|). \quad (4.9)$$

First we substitute

$$x = \hat{Y}_{t_\varepsilon} - y = \varepsilon f(t_\varepsilon, y, z, u) + \varepsilon \int_E u(e)n(de)$$

in the definition of $\alpha(\cdot)$. This provides us

$$E[d_K^2(\hat{Y}_{t_\varepsilon})|\mathcal{F}_{t_*}] = d_K^2(y) + \varepsilon E[\langle \nabla d_K^2(y), f(t_\varepsilon, y, z, u) + \int_E u(e)n(de) \rangle |\mathcal{F}_{t_*}] + \varepsilon \gamma_\varepsilon^1,$$

where

$$\begin{aligned} \gamma_\varepsilon^1 := & \frac{\varepsilon}{2} E[\langle D^2 d_K^2(y)(f(t_\varepsilon, y, z, u) + \int_E u(e)n(de)), f(t_\varepsilon, y, z, u) + \int_E u(e)n(de) \rangle \\ & + \frac{2}{\varepsilon^2} \alpha(\varepsilon f(t_\varepsilon, y, z, u) + \varepsilon \int_E u(e)n(de)) |\mathcal{F}_{t_*}]. \end{aligned}$$

By (H3), $f(t_\varepsilon, y, z, u) + \int_E u(e)n(de)$ can be dominated by some nonnegative random variable which belongs to $L^2(\Omega, \mathcal{F}_{t_*}, P)$, so there exists some nonnegative

random variable $\zeta \in L^1(\Omega, \mathcal{F}_{t_*}, P)$ whose norm depends only on y, z and u such that

$$|\gamma_\varepsilon^1| \leq \zeta \varepsilon + E\left[\frac{\alpha(x)}{|x|^2} \varepsilon |f(t_\varepsilon, y, z, u) + \int_E u(e)n(de)|^2 | \mathcal{F}_{t_*}]\right].$$

Hence from Lebesgue's dominated convergence theorem, $\gamma_\varepsilon^1 \rightarrow 0, P - a.s.$, as ε tends to 0.

We now substitute

$$x = \xi - y = z(W_t - W_{t_\varepsilon}) + \int_{t_\varepsilon}^t \int_E u(e)N(drde)$$

in the definition of $\alpha(\cdot)$. If we set

$$\xi_s = y + z(W_s - W_{t_\varepsilon}) + \int_{t_\varepsilon}^s \int_E u(e)N(drde), s \in [t_\varepsilon, t],$$

we have

$$\begin{aligned} & E[d_K^2(\xi) - d_K^2(y) | \mathcal{F}_{t_*}] \\ = & E\left\{ \int_{t_\varepsilon}^t \int_E [d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})] N(dsde) \right. \\ & \left. + \frac{1}{2} \langle D^2 d_K^2(y) z(W_t - W_{t_\varepsilon}), z(W_t - W_{t_\varepsilon}) \rangle + \alpha(z(W_t - W_{t_\varepsilon})) | \mathcal{F}_{t_*} \right\} \\ = & \varepsilon E\left\{ \frac{1}{\varepsilon} \int_{t_\varepsilon}^t \int_E [d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})] n(de) ds + \frac{1}{2} \langle D^2 d_K^2(y) z, z \rangle | \mathcal{F}_{t_*} \right\} + \varepsilon \gamma_\varepsilon^2, \end{aligned}$$

where

$$\gamma_\varepsilon^2 = \frac{1}{\varepsilon} E[\alpha(z(W_t - W_{t_\varepsilon})) | \mathcal{F}_{t_*}] = \frac{1}{\varepsilon} E[\alpha(\sqrt{\varepsilon} z W_1)]$$

is such that

$$\limsup_{\varepsilon \rightarrow 0} \gamma_\varepsilon^2 \leq 0.$$

In fact, on one hand,

$$\frac{1}{\varepsilon} \alpha(\sqrt{\varepsilon} z W_1) \rightarrow 0, P - a.s., \text{ as } \varepsilon \rightarrow 0.$$

And on the other hand, for $\varepsilon > 0$, with (4.9),

$$\frac{1}{\varepsilon} \alpha(\sqrt{\varepsilon} z W_1) \leq (1 + |D^2 d_K^2(y)|) |z|^2 |W_1|^2 \in L^1(P).$$

Finally, we get from Fatou's Lemma

$$\limsup_{\varepsilon \rightarrow 0} \gamma_\varepsilon^2 \leq E[\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \alpha(\sqrt{\varepsilon} z W_1) | \mathcal{F}_{t_*}] = 0, P - a.s..$$

Therefore

$$\begin{aligned}
& E[d_K^2(\hat{Y}_{t_\varepsilon}) - d_K^2(\xi)|\mathcal{F}_{t_*}] \\
&= \varepsilon \{ E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e)n(de) \rangle |\mathcal{F}_{t_*}] \\
&\quad + E[\langle \nabla d_K^2(y), f(t_\varepsilon, y, z, u) - f(t, y, z, u) \rangle |\mathcal{F}_{t_*}] - \frac{1}{2} \langle D^2[d_K^2(y)]z, z \rangle \\
&\quad - \frac{1}{\varepsilon} E[\int_{t_\varepsilon}^t \int_E (d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-}))n(de)ds |\mathcal{F}_{t_*}] + \gamma_\varepsilon^1 - \gamma_\varepsilon^2 \} \\
&\geq \varepsilon \{ E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e)n(de) \rangle |\mathcal{F}_{t_*}] - \frac{1}{2} \langle D^2[d_K^2(y)]z, z \rangle \\
&\quad - \frac{1}{\varepsilon} E[\int_{t_\varepsilon}^t \int_E (d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-}))n(de)ds |\mathcal{F}_{t_*}] + \gamma_\varepsilon^1 - \gamma_\varepsilon^2 - \gamma_\varepsilon^3 \},
\end{aligned}$$

where $\gamma_\varepsilon^3 = E[\langle \nabla d_K^2(y), |f(t_\varepsilon, y, z, u) - f(t, y, z, u)| \rangle |\mathcal{F}_{t_*}]$. By (H1) and (H3), we have

$$\lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^3 \rightarrow 0, P - a.s..$$

So the proof of Lemma 4.2 is completed by setting $\gamma_\varepsilon = \gamma_\varepsilon^1 - \gamma_\varepsilon^2 - \gamma_\varepsilon^3$.

□

Note that due to the Lemma 4.1 and 4.2:

$$\begin{aligned}
& E[d_K^2(Y_{t_\varepsilon}) - d_K^2(\xi)|\mathcal{F}_{t_*}] \\
&\geq \varepsilon \{ E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e)n(de) \rangle |\mathcal{F}_{t_*}] \\
&\quad - \frac{1}{2} \langle D^2[d_K^2(y)]z, z \rangle - \frac{1}{\varepsilon} E[\int_{t_\varepsilon}^t \int_E (d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-}))n(de)ds |\mathcal{F}_{t_*}] + \gamma_\varepsilon \},
\end{aligned}$$

for some $\gamma_\varepsilon = \gamma_\varepsilon(y, z, u)$ such that $\liminf_{\varepsilon \rightarrow 0} \gamma_\varepsilon \geq 0, P - a.s..$

Let us return to the proof of the necessity. For this denote by $(\tilde{Y}, \tilde{Z}, \tilde{U})$ the unique solution to the following BSDE with jump:

$$\tilde{Y}_s = \eta + \int_s^t f(r, \tilde{Y}_r, \tilde{Z}_r, \tilde{U}_r)dr - \int_s^t \tilde{Z}_r dW_r - \int_s^t \int_E \tilde{U}_r(e) \tilde{N}(drde), s \in [t_\varepsilon, t],$$

where $\eta \in L^2(\Omega, \mathcal{F}, P)$ is a measurable selection of the set

$$\{(\omega, x) \in \Omega \times R^m | x \in \Pi_K(\xi(\omega))\} \in \mathcal{F}_t \otimes \mathcal{B}(R^m).$$

We assume that K enjoys the BSVP. Hence $\tilde{Y}_s \in K$, for $t_\varepsilon \leq s \leq t, P - a.s..$ This implies

$$0 \geq E[d_K^2(Y_{t_\varepsilon}) - |Y_{t_\varepsilon} - \tilde{Y}_{t_\varepsilon}|^2 |\mathcal{F}_{t_*}] = E[d_K^2(Y_{t_\varepsilon}) - d_K^2(\xi)|\mathcal{F}_{t_*}] - E[|Y_{t_\varepsilon} - \tilde{Y}_{t_\varepsilon}|^2 - |\xi - \eta|^2 |\mathcal{F}_{t_*}].$$

From Itô's formula,

$$\begin{aligned}
& E[|Y_{t_\varepsilon} - \tilde{Y}_{t_\varepsilon}|^2 - |\xi - \eta|^2 | \mathcal{F}_{t_*}] \\
&= 2E\left[\int_{t_\varepsilon}^t \langle Y_s - \tilde{Y}_s, f(s, Y_s, Z_s, U_s) - f(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \rangle ds \middle| \mathcal{F}_{t_*}\right] \\
&\quad - E\left[\int_{t_\varepsilon}^t |Z_s - \tilde{Z}_s|^2 ds + \int_{t_\varepsilon}^t \int_E |U_s - \tilde{U}_s|^2 n(de) ds \middle| \mathcal{F}_{t_*}\right] \\
&\leq C \int_{t_\varepsilon}^t E[|Y_s - \tilde{Y}_s|^2 | \mathcal{F}_{t_*}] ds,
\end{aligned}$$

where $C = 2L + 2L^2$. Consequently

$$\begin{aligned}
0 &\geq \frac{1}{\varepsilon} E[d_K^2(Y_{t_\varepsilon}) - d_K^2(\xi) | \mathcal{F}_{t_*}] - \frac{C}{\varepsilon} \int_{t_\varepsilon}^t E[|Y_s - \tilde{Y}_s|^2 | \mathcal{F}_{t_*}] ds \\
&\geq E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e) n(de) \rangle | \mathcal{F}_{t_*}] - \frac{1}{2} \langle D^2[d_K^2(y)] z, z \rangle + \gamma_\varepsilon \\
&\quad - \frac{1}{\varepsilon} E\left[\int_{t_\varepsilon}^t \int_E (d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})) n(de) ds \middle| \mathcal{F}_{t_*}\right] - \frac{C}{\varepsilon} \int_{t_\varepsilon}^t E[|Y_s - \tilde{Y}_s|^2 | \mathcal{F}_{t_*}] ds.
\end{aligned}$$

In order to estimate the integral term in the last estimate, note that

$$\begin{aligned}
& E[|Y_s - \tilde{Y}_s|^2 | \mathcal{F}_{t_*}] \\
&= E[|(Y_s - \xi) - (\tilde{Y}_s - \eta) + (\xi - \eta)|^2 | \mathcal{F}_{t_*}] \\
&\leq E[d_K^2(\xi) | \mathcal{F}_{t_*}] + 2E[|Y_s - \xi|^2 + |\tilde{Y}_s - \eta|^2 | \mathcal{F}_{t_*}] \\
&\quad + 2(E[d_K^2(\xi) | \mathcal{F}_{t_*}])^{\frac{1}{2}} \{ (E[|Y_s - \xi|^2 | \mathcal{F}_{t_*}])^{\frac{1}{2}} + (E[|\tilde{Y}_s - \eta|^2 | \mathcal{F}_{t_*}])^{\frac{1}{2}} \}.
\end{aligned} \tag{4.10}$$

While

$$\begin{aligned}
& |Y_s - \xi|^2 + |\tilde{Y}_s - \eta|^2 \\
&\leq 2(|Y_s - E[\xi | \mathcal{F}_s]|^2 + |\xi - E[\xi | \mathcal{F}_s]|^2 + |\tilde{Y}_s - E[\eta | \mathcal{F}_s]|^2 + |\eta - E[\eta | \mathcal{F}_s]|^2),
\end{aligned}$$

so from Proposition 2.4 we conclude

$$\sup_{s \in [t_\varepsilon, t]} E[|Y_s - \xi|^2 + |\tilde{Y}_s - \eta|^2 | \mathcal{F}_{t_*}] \rightarrow 0, P - a.s.,$$

as ε tends to 0. On the other hand,

$$\begin{aligned}
& E[d_K^2(\xi) - d_K^2(y) | \mathcal{F}_{t_*}] \\
&= E\left\{ \int_{t_\varepsilon}^t \int_E [d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})] N(ds de) + d_K^2(y + z(W_t - W_{t_\varepsilon})) - d_K^2(y) \middle| \mathcal{F}_{t_*} \right\}.
\end{aligned}$$

Recall that $\xi_s = y + z(W_s - W_{t_\varepsilon}) + \int_{t_\varepsilon}^s \int_E u(e) N(dr de)$, $s \in [t_\varepsilon, t]$, so

$$\xi_{s-} \rightarrow y, P - a.s., \quad \text{as } \varepsilon \rightarrow 0.$$

By the Lebesgue theorem of dominated convergence, when ε tends to 0,

$$\begin{aligned} & \frac{1}{\varepsilon} E \left\{ \int_{t_\varepsilon}^t \int_E [d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})] N(dsde) | \mathcal{F}_{t_*} \right\} \\ & \rightarrow \int_E [d_K^2(y + u(e)) - d_K^2(y)] n(de), P - a.s.. \end{aligned} \quad (4.11)$$

Thus

$$E \left\{ \int_{t_\varepsilon}^t \int_E [d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})] N(dsde) | \mathcal{F}_{t_*} \right\} \rightarrow 0, P - a.s., \text{ as } \varepsilon \rightarrow 0.$$

Obviously, the function d_K^2 is continuous at y , using again Lebesgue's dominated convergence theorem, we have

$$E[d_K^2(y + z(W_t - W_{t_\varepsilon})) - d_K^2(y) | \mathcal{F}_{t_*}] \rightarrow 0, P - a.s., \text{ as } \varepsilon \rightarrow 0.$$

So we conclude that

$$E[d_K^2(\xi) - d_K^2(y) | \mathcal{F}_{t_*}] \rightarrow 0, P - a.s., \text{ as } \varepsilon \rightarrow 0.$$

Consequently, from (4.10) and above, we have

$$E[|Y_s - \tilde{Y}_s|^2 | \mathcal{F}_{t_*}] \leq d_K^2(y) + \beta_\varepsilon^2, s \in [t_\varepsilon, t],$$

where β_ε^2 converges to 0, $P - a.s.$, as ε tends to 0.

Therefore

$$\frac{C}{\varepsilon} \int_{t_\varepsilon}^t E[|Y_s - \tilde{Y}_s|^2 | \mathcal{F}_{t_*}] ds \leq C(d_K^2(y) + \beta_\varepsilon^2)$$

and

$$\begin{aligned} & E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e) n(de) \rangle | \mathcal{F}_{t_*}] - \frac{1}{2} \langle D^2[d_K^2(y)] z, z \rangle \\ & - \frac{1}{\varepsilon} E \left[\int_{t_\varepsilon}^t \int_E (d_K^2(\xi_{s-} + u(e)) - d_K^2(\xi_{s-})) n(de) ds | \mathcal{F}_{t_*} \right] - C d_K^2(y) + \gamma_\varepsilon - C \beta_\varepsilon^2 \leq 0, P - a.s.. \end{aligned}$$

Finally, since $\liminf_{\varepsilon \rightarrow 0} (\gamma_\varepsilon - C \beta_\varepsilon^2) \geq 0, P - a.s..$ This with (4.11), let ε tend to 0, we have

$$\begin{aligned} & E[\langle \nabla d_K^2(y), f(t, y, z, u) + \int_E u(e) n(de) \rangle | \mathcal{F}_{t_*}] \\ & - \frac{1}{2} \langle D^2[d_K^2(y)] z, z \rangle - \int_E [d_K^2(y + u(e)) - d_K^2(y)] n(de) - C d_K^2(y) \leq 0, \end{aligned}$$

P -almost everywhere, for $t_* \in [0, T]$. Passing to the limit $t_* \rightarrow t$, we obtain the wished result if we choose $C^* = 2C$ and note that

$$\nabla d_K^2(y) = 2(y - \Pi_K(y)).$$

□

(b) Sufficiency. Let K be a nonempty convex closed subset of R^m . Suppose that (2.6) holds true. Let $\eta \in C^\infty(R^m)$ be a nonnegative function with support in the unit ball and such that

$$\int_{R^m} \eta(x) dx = 1.$$

For $\delta > 0$, we put

$$\begin{aligned} \eta_\delta(x) &:= \frac{1}{\delta^m} \eta\left(\frac{x}{\delta}\right) \\ \phi_\delta(x) &:= d_K^2 \star \eta_\delta(x) := \int_{R^m} d_K^2(x - x') \eta_\delta(x') dx' = \int_{R^m} d_K^2(x') \eta_\delta(x - x') dx', x \in R^m. \end{aligned}$$

Obviously, $\phi_\delta \in C^\infty(R^m)$. We can see from [7] that the function ϕ_δ satisfies the following properties

$$\left\{ \begin{array}{ll} \text{(i)} & 0 \leq \phi_\delta(x) \leq (d_K(x) + \delta)^2, \\ \text{(ii)} & \nabla \phi_\delta(x) = \int_{R^m} (\nabla d_K^2)(x') \eta_\delta(x - x') dx', \\ & |\nabla \phi_\delta(x)| \leq 2(d_K(x) + \delta), \\ \text{(iii)} & D^2 \phi_\delta(x) = \int_{R^m} D^2[d_K^2](x') \eta_\delta(x - x') dx', \\ & 0 \leq D^2 \phi_\delta(x) \leq 2I, \end{array} \right. \quad (4.12)$$

for all $x \in R^m$.

Consider $\xi \in L^2(\Omega, \mathcal{F}_T, P, K)$ and let (Y, Z, U) be the unique solution to the following BSDE with jump

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \tilde{N}(ds de), \quad t \in [0, T].$$

(4.12) enables us to apply Itô's formula to $\phi_\delta(Y_t)$ and to deduce that, for $0 \leq t \leq$

$T, \delta > 0$,

$$\begin{aligned}
& E\phi_\delta(Y_t) \\
&= E\phi_\delta(\xi) + E \int_t^T \langle (\nabla \phi_\delta)(Y_s), f(s, Y_s, Z_s, U_s) \rangle ds - \frac{1}{2} E \int_t^T \langle (D^2 \phi_\delta)(Y_s) Z_s, Z_s \rangle ds \\
&\quad - E \int_t^T \int_E [\phi_\delta(Y_s + U_s(e)) - \phi_\delta(Y_s) - \langle \nabla \phi_\delta(Y_s), U_s(e) \rangle] n(de) ds \\
&\leq \delta^2 - E \int_t^T \int_{R^m} \{ \langle \nabla d_K^2(y), f(s, y, Z_s, U_s) - f(s, Y_s, Z_s, U_s) \rangle \} \eta_\delta(Y_s - y) dy ds \\
&\quad + E \int_t^T \int_{R^m} \{ \langle \nabla d_K^2(y), f(s, y, Z_s, U_s) \rangle - \frac{1}{2} \langle D^2(d_K^2(y)) Z_s, Z_s \rangle \\
&\quad \quad - \int_E [d_K^2(y + U_s(e)) - d_K^2(y) - \langle \nabla d_K^2(y), U_s(e) \rangle] n(de) \} \eta_\delta(Y_s - y) dy ds.
\end{aligned}$$

Then from (2.6) and (4.12), for $\delta \in (0, 1)$,

$$\begin{aligned}
& E\phi_\delta(Y_t) \\
&\leq \delta^2 + \frac{C^*}{2} E \int_t^T \int_{R^m} d_K^2(y) \eta_\delta(Y_s - y) dy ds \\
&\quad + E \int_t^T \int_{R^m} 2d_K(y) \max_{y: |y-Y_s| \leq \delta} |f(s, y, Z_s, U_s) - f(s, Y_s, Z_s, U_s)| \eta_\delta(Y_s - y) dy ds \\
&\leq \delta^2 + \frac{C^*}{2} \int_t^T E[\phi_\delta(Y_s)] ds \\
&\quad + E \int_t^T (1 + \phi_\delta(Y_s)) \max_{y: |y-Y_s| \leq \delta} |f(s, y, Z_s, U_s) - f(s, Y_s, Z_s, U_s)| ds.
\end{aligned}$$

Since

$$\begin{aligned}
& E \int_t^T \phi_\delta(Y_s) \max_{y: |y-Y_s| \leq \delta} |f(s, y, Z_s, U_s) - f(s, Y_s, Z_s, U_s)| ds \\
&\leq E \int_t^T \phi_\delta(Y_s) \max_{y: |y-Y_s| \leq \delta} L|Y_s - y| ds \\
&\leq E \int_t^T L\delta \phi_\delta(Y_s) ds,
\end{aligned}$$

We choose $\delta < \frac{1}{L}$, then

$$E\phi_\delta(Y_t) \leq \delta^2 + L(T-t)\delta + (1 + \frac{C^*}{2}) \int_t^T E\phi_\delta(Y_s) ds. \quad (4.13)$$

for $0 \leq t \leq T$. On the other hand, from (4.12-i), we deduce

$$E\phi_\delta(Y_t) \leq E[d_K(Y_t) + \delta]^2 \leq 2E(d_K^2(Y_t) + \delta^2) \leq 2E(2|Y_t|^2 + 2d_K^2(0) + \delta^2) < +\infty.$$

This allows to apply Gronwall's inequality to (4.13), it yields

$$E\phi_\delta(Y_t) \leq [\delta^2 + L(T-t)\delta]e^{(1+\frac{C^*}{2})T}.$$

Note that

$$\phi_\delta(Y_t) = \int_{R^m} d_K^2(Y_t - y)\eta_\delta(y)dy \rightarrow d_K^2(Y_t), \quad \text{as } \delta \rightarrow 0, P - a.s..$$

Thus, from Fatou's Lemma, we conclude that

$$Ed_K^2(Y_t) = E[\lim_{\delta \rightarrow 0} \phi_\delta(Y_t)] \leq \liminf_{\delta \rightarrow 0} E\phi_\delta(Y_t) \leq \liminf_{\delta \rightarrow 0} [\delta^2 + L(T-t)\delta]e^{(1+\frac{C^*}{2})T} = 0,$$

for $0 \leq t \leq T$. That is

$$P\{\omega : Y_t(\omega) \in K\} = 1, \quad \forall t \in [0, T]$$

which is equivalent to

$$P\{\omega : Y_t(\omega) \in K, \forall t \in [0, T]\} = 1.$$

□

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